

A PRODUCT THEOREM FOR A GENERAL ACCUMULATION PROPERTY

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ABSTRACT. We characterize the situations in which certain accumulation properties of topological spaces are preserved under taking products.

1. INTRODUCTION

In [MG] we introduced the following general notion of accumulation relative to some set E , and proved that it is connected with covering properties. If I is a set, $\mathcal{S}(I)$ denotes the set of all subsets of I .

Definition 1.1. Let I be a set, E be a subset of $\mathcal{S}(I)$, and $(x_i)_{i \in I}$ be an I -indexed sequence of elements of some topological space X .

We say that a point $x \in X$ is an *accumulation point in the sense of E* , or simply an *E -accumulation point*, of the sequence $(x_i)_{i \in I}$ if and only if, for every open neighborhood U of x , the set $\{i \in I \mid x_i \in U\}$ belongs to E .

We say that X satisfies the *E -accumulation property* (or that X is *E -compact*) if and only if every I -indexed sequence of elements of X has some (not necessarily unique) accumulation point in the sense of E .

The definition is motivated by a number of examples.

As the simplest possible non trivial case, if $I = \omega$, and E is the set of all cofinite subsets of ω , then x is an E -accumulation point of $(x_n)_{n \in \omega}$ in the sense of Definition 1.1 if and only if $(x_n)_{n \in \omega}$ converges to x .

As a slightly more involved example, if $E = D$ is an ultrafilter over I , then x is a D -accumulation point of the sequence $(x_i)_{i \in I}$ if and only if $(x_i)_{i \in I}$ *D -converges* to x . Thus, in the particular case of an ultrafilter, the D -accumulation property is the usual notion of *D -compactness*.

In the above examples E is a filter, a condition which, as we shall show, will turn out to provide a dividing line between very different situations. We now shall provide an example in which E is not a filter.

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Example 1.2. Suppose that $|I| = \lambda$ is an infinite regular cardinal, and E is the set of all subsets of I having cardinality λ . In this case the E -accumulation property turns out to be equivalent to a property usually called CAP_λ . A topological space X is said to satisfy CAP_λ if and only if every subset $Y \subseteq X$ of cardinality λ has some *complete accumulation point*, that is, a point $x \in X$ such that $|Y \cap U| = \lambda$, for every neighborhood U of x in X .

See [MG] for a complete discussion about E -accumulation and its connection with other compactness properties.

In this paper we discuss conditions under which the E -accumulation property is productive (that is, preserved under taking arbitrary products).

We assume no separation axiom. If not otherwise specified, when we speak of a product of a family of topological spaces, we always mean the Tychonoff product.

2. PRODUCTIVITY

As usual, by a *filter* on some set I , we mean a *nonempty proper* subset E of $\mathcal{S}(I)$ which is closed under finite intersections and under taking supersets.

Proposition 2.1. *Let I, J be sets, E be a subset of $\mathcal{S}(I)$, and suppose that $(X_j)_{j \in J}$ is a sequence of topological spaces. Consider the following conditions.*

- (1) *The Tychonoff product $\prod_{j \in J} X_j$ satisfies the E -accumulation property.*
- (2) *For every $j \in J$, X_j satisfies the E -accumulation property.*

Condition (1) implies Condition (2).

If E is a filter, then Condition (2) implies Condition (1).

In fact, a local version of the result holds. Proposition 2.1 is immediate from the following lemma.

Lemma 2.2. *Let I, J be sets, E be a subset of $\mathcal{S}(I)$, and suppose that $(X_j)_{j \in J}$ is a sequence of topological spaces. Suppose that $(x_{j,i})_{j \in J, i \in I}$ is an I -indexed sequence of elements of $\prod_{j \in J} X_j$, and that $z = (z_j)_{j \in J}$ is an element of $\prod_{j \in J} X_j$. If*

- (1) *z is an E -accumulation point of $(x_{j,i})_{j \in J, i \in I}$ in $\prod_{j \in J} X_j$,*

then

- (2) *for every $\bar{j} \in J$, the element $z_{\bar{j}} \in X_{\bar{j}}$ is an E -accumulation point of the I -indexed sequence $(x_{\bar{j},i})_{i \in I}$ of elements of $X_{\bar{j}}$.*

If E is a filter, then (2) implies (1).

Proof. Suppose that (1) holds. For each $\bar{j} \in J$, and for every open neighborhood U of $z_{\bar{j}}$ in $X_{\bar{j}}$, consider the open neighborhood V of z in $\prod_{j \in J} X_j$ defined by $V = \prod_{j \in J} V_j$, where $V_j = X_j$, if $j \neq \bar{j}$, and $V_{\bar{j}} = U$.

Then $\{i \in I \mid x_{\bar{j},i} \in U \text{ in } X_{\bar{j}}\} = \{i \in I \mid (x_{j,i})_{j \in J} \in V \text{ in } \prod_{j \in J} X_j\} \in E$, since z is an E -accumulation point of $(x_{j,i})_{j \in J, i \in I}$ in $\prod_{j \in J} X_j$.

Conversely, suppose that E is a filter, and that (2) holds. Let U be an open neighborhood of z in $\prod_{j \in J} X_j$. Then there is an open neighborhood U' of z such that $U' \subseteq U$ and $U' = \prod_{j \in J} U_j$, where, for every $j \in J$, U_j is an open neighborhood of z_j in X_j , and there is some finite $F \subseteq J$ such that $U_j = X_j$, for every $j \in J \setminus F$.

By (2), for every $j \in F$, $e_j = \{i \in I \mid x_{j,i} \in U_j\} \in E$, hence $\{i \in I \mid (x_{j,i})_{j \in J} \in U'\} = \bigcap_{j \in F} e_j \in E$, since E is closed under finite intersection. Since $U' \subseteq U$, then $\{i \in I \mid (x_{j,i})_{j \in J} \in U'\} \subseteq \{i \in I \mid (x_{j,i})_{j \in J} \in U\} \in E$, since E is closed under supersets.

Since U was an arbitrary open neighborhood of z in $\prod_{j \in J} X_j$, the above argument shows that z is an E -accumulation point of $(x_{j,i})_{j \in J, i \in I}$ in $\prod_{j \in J} X_j$. \square

As the example of complete accumulation points shows, restricting Definition 1.1 only to filters causes a big loss in generality. However we can show that, in some local sense, we get an equivalent definition when we let E vary only among filters. This fact is the main key for applications to products.

Lemma 2.3. *Suppose that I is a set, E is a subset of $\mathcal{S}(I)$, and X is a topological space. Let $(x_i)_{i \in I}$ be some given I -indexed sequence of elements of X , and suppose that $x \in X$ is an E -accumulation point of $(x_i)_{i \in I}$.*

If, for every neighborhood U of x in X , we put $f_U = \{i \in I \mid x_i \in U\}$, then $F = \{f_U \mid U \text{ is an open neighborhood of } x \text{ in } X\}$ is closed under finite intersections, $F \subseteq E$, and x is an F -accumulation point of $(x_i)_{i \in I}$.

Proof. That $F \subseteq E$ is immediate from Definition 1.1, and the assumption that x is an E -accumulation point of $(x_i)_{i \in I}$.

If U and V are open neighborhoods of x , then $U \cap V$ is an open neighborhood of x , and $f_U \cap f_V = f_{U \cap V} \in F$, hence F is closed under intersections.

By the definition of F , x is an F -accumulation point of $(x_i)_{i \in I}$. \square

Lemma 2.3 may be seen as a general version of the fact that an element $x \in X$ belongs to the closure of some set $Y \subseteq X$ if and only if

there is some net of elements of Y converging to x [E, Proposition 1.6.3], or, in equivalent form, if and only if there exists a filter-base consisting of subsets of Y converging to x [E, Proposition 1.6.9]. Indeed, this latter result is a consequence of the particular case of Lemma 2.3 when $I = Y$ and $E = \mathcal{S}(I) \setminus \{\emptyset\}$.

In a rough but suggestive sense, Lemma 2.3 asserts that, locally, “accumulation” becomes a form of “convergence” (an arbitrary E corresponds to some notion of accumulation, but a *filter* F corresponds to a notion of convergence). Since most notions of convergence are productive (the last statement in Proposition 2.1, as far as we are concerned), and since any failure of accumulation implies some local failure, we get that an *accumulation* property is productive, for some given space X , if and only if X satisfies some stronger notion of *convergence*.

Theorem 2.4. *Suppose that X is a topological space, I is a set, and E is a subset of $\mathcal{S}(I)$ closed under taking supersets. Then the following conditions are equivalent.*

- (1) *Every power of X satisfies the E -accumulation property.*
- (2) *X satisfies the F -accumulation property, for some filter $F \subseteq E$.*
- (3) *X satisfies the F -accumulation property, for some filter F which is maximal among the filters contained in E .*
- (4) *X^δ satisfies the E -accumulation property, where δ is the cardinality of the set of all the filters which are maximal among the filters contained in E .*
- (5) *X^δ satisfies the E -accumulation property, for $\delta = |X|^{|I|}$.*

Proof. (1) \Rightarrow (4), (1) \Rightarrow (5) and (3) \Rightarrow (2) are trivial. We shall complete the proof by proving (2) \Rightarrow (1), (4) \Rightarrow (3) and (5) \Rightarrow (2).

(2) \Rightarrow (1) By the last statement in Proposition 2.1, every power of X satisfies the F -accumulation property. Since $F \subseteq E$, the E -accumulation property follows trivially.

In order to prove that (4) implies (3), we shall suppose that (4) holds and (3) fails, and we shall reach a contradiction. If (3) fails, then, for every maximal filter $F \subseteq E$, there is a sequence $(x_i)_{i \in I}$ such that no $x \in X$ is an F -accumulation point of $(x_i)_{i \in I}$.

Enumerate all maximal filters $\subseteq E$ as $(F_j)_{j \in \delta}$, and, for each $j \in \delta$, choose a sequence $(x_{j,i})_{i \in I}$ which has no F_j -accumulation point in X .

Consider $(x_{j,i})_{j \in \delta, i \in I}$ as a single I -indexed sequence of elements of X^δ . By (4), $(x_{j,i})_{j \in \delta, i \in I}$ has some E -accumulation point $z = (z_j)_{j \in \delta}$ in X^δ . By applying Lemma 2.3 to $(x_{j,i})_{j \in \delta, i \in I}$ and z in X^δ , we get some $F' \subseteq E$ such that F' is closed under finite intersections, and z is an F' -accumulation point of $(x_{j,i})_{j \in \delta, i \in I}$. If F'' is the filter generated by F' , then also $F'' \subseteq E$, since E is closed under taking supersets. By an easy

application of Zorn's Lemma, there is a filter F such that $F'' \subseteq F \subseteq E$, and F is maximal among the filters contained in E . Since $F' \subseteq F$, then z is an F -accumulation point of $(x_{j,i})_{j \in \delta, i \in I}$.

By the construction of the sequence $(F_j)_{j \in \delta}$, there is $\bar{j} \in \delta$ such that $F = F_{\bar{j}}$. By Lemma 2.2 (1) \Rightarrow (2), $z_{\bar{j}}$ is an F -accumulation point of $(x_{\bar{j},i})_{i \in I}$ in X . This contradicts the choice of the sequence $(x_{\bar{j},i})_{i \in I}$. We have reached a contradiction, thus (4) \Rightarrow (3) is proved.

The proof that (5) implies (2) is rather similar, and somewhat simpler. We shall suppose that (5) holds and that (2) fails, and we shall reach a contradiction. This time, enumerate all I -indexed sequences of elements of X , as $(x_{j,i})_{j \in \delta, i \in I}$. This is possible since there are exactly δ -many such sequences.

By (5), the single sequence $(x_{j,i})_{j \in \delta, i \in I}$ in X^δ has some E -accumulation point $z = (z_j)_{j \in \delta}$ in X^δ . By Lemma 2.3, and since E is closed under supersets, there is a filter $F \subseteq E$ such that z is an F -accumulation point of $(x_{j,i})_{j \in \delta, i \in I}$. Since (2) fails, there is a sequence $(x_i)_{i \in I}$ such that no $x \in X$ is an F -accumulation point of $(x_i)_{i \in I}$. Since $(x_{j,i})_{j \in \delta, i \in I}$ enumerates all possible I -indexed sequences of elements of X , there is $\bar{j} \in \delta$ such that $x_{\bar{j},i} = x_i$, for every $i \in I$, thus $(x_{\bar{j},i})_{i \in I}$ witnesses the failure of the F -accumulation property for X . However, Lemma 2.2 (1) \Rightarrow (2), with F in place of E , implies that $z_{\bar{j}}$ is an F -accumulation point of $(x_{\bar{j},i})_{i \in I}$, a contradiction. \square

By Proposition 3.2 in Section 3 below, if Condition (2) in Theorem 2.4 holds, then F is actually an ultrafilter over I , except for very special cases. This fact might explain the necessity of the use of ultrafilters in many product theorems.

The simplest case in Theorem 2.4 is of course when E itself is a filter, in which case we get that the E -accumulation property is productive, a fact which follows already from Proposition 2.1.

The case of complete accumulation points is more significant. As a corollary of Theorem 2.4, we get a result by Ginsburg and Saks [GS, Sa], asserting that, for every topological space X , every power of X satisfies CAP_λ if and only if X is D -compact, for some ultrafilter D uniform over λ . We shall prove a slightly more general version which deals with ordinals. It is not completely clear how much significant this improvement is, however we include it, since its proof involves no new technical detail.

Definition 2.5. If α is an ordinal, X is a topological space, and $(x_\gamma)_{\gamma \in \alpha}$ is a sequence of elements of X , we say that $x \in X$ is an α -complete accumulation point of $(x_\gamma)_{\gamma \in \alpha}$ if and only if, for every neighborhood U of x , the set $\{\gamma \in \alpha \mid x_\gamma \in U\}$ has order type α .

We say that X satisfies the α -complete accumulation property, CAP_α^* for short, if and only if every α -indexed sequence $(x_\gamma)_{\gamma \in \alpha}$ of elements of X has some α -complete accumulation point in X .

It is easy to see that, if α is a *regular* cardinal, then the starred CAP_α^* turns out to be equivalent to the unstarred CAP_α of Example 1.2. If α is a singular cardinal, then CAP_α^* is equivalent to the conjunction of CAP_α and $\text{CAP}_{\text{cf } \alpha}$. See [L, Proposition 3.3].

Remark 2.6. Notice that the notions in the above definition are the particular cases of Definition 1.1 when $I = \alpha$ and E is the set of all subsets of α having order type α .

Definition 2.7. As usual, a filter F over some set I is said to be *uniform* if and only if every element of F has cardinality $|I|$. We generalize this notion to ordinals.

If α is an ordinal, we say that a filter F over α is *ordinal-uniform* if and only if every member of F has order type α .

Remark 2.8. If α is an infinite cardinal, then F over α is uniform if and only if it is ordinal-uniform. When α is not a cardinal, the two notions are distinct. Ordinal-uniformity implies uniformity, but the converse does not necessarily hold.

As far as ordinal-uniformity is concerned, there is a strong dichotomy according to the form of the ordinal on which a filter is taken.

Proposition 2.9. (1) If $\alpha = \omega^\eta$ (ordinal exponentiation), for some ordinal η (in particular, when α is an infinite cardinal), then any filter ordinal-uniform over α can be extended to an ultrafilter ordinal-uniform over α .

(2) If α has not the form ω^η , for some η , then there is no ordinal-uniform ultrafilter over α .

Proof. (1) We show that if $\alpha = \omega^\eta$, F is a filter, every element of F has order type α , and $G = \{Z \subseteq \alpha \mid \alpha \setminus Z \text{ has order type } < \alpha\}$, then $F \cup G$ has the finite intersection property. Indeed, due to the particular form of α , the union of a finite number of subsets of α of order type $< \alpha$ has order type $< \alpha$, hence G is closed under finite intersections. Moreover, the intersection of an element of F and an element of G is nonempty (in this case, the particular form of α plays no role). Hence $F \cup G$ can be extended to an ultrafilter D over α , and D is ordinal-uniform, since D extends G .

(2) There are disjoint $Z_1, Z_2 \subseteq \alpha$ such that $Z_1 \cup Z_2 = \alpha$, and both Z_1 and Z_2 have order type $< \alpha$ (just express α in normal form, and consider it as a sum of two summands). Since any ultrafilter D over α must contain one of Z_1 or Z_2 , then D is not ordinal-uniform. \square

Corollary 2.10. *For every topological space X and every ordinal α , the following conditions are equivalent.*

- (1) *Every power of X satisfies CAP_α^* .*
- (2) *X is F -compact, for some filter F ordinal-uniform over α .*
- (3) *X^δ satisfies CAP_α^* , for $\delta = \min\{2^{2^{|\alpha|}}, |X|^{|\alpha|}\}$.*

If, in addition, $\alpha = \omega^\eta$ (ordinal exponentiation), for some ordinal η , then the preceding conditions are also equivalent to:

- (4) *X is D -compact, for some ultrafilter D ordinal-uniform over α .*

Proof. The Corollary is immediate from Theorem 2.4, by taking E to be the set of all subsets of α of order type α , and using Remarks 2.6 and 2.8 and Proposition 2.9. \square

The case $\alpha = \omega$ of Corollary 2.10 is due to Ginsburg and Saks [GS, Theorem 2.6], via the equivalence between CAP_ω^* and countable compactness. For α an arbitrary *cardinal*, the equivalence of (1) and (2) in Corollary 2.10 appears in [Sa, Theorem 6.2]. Related results appear in [SS, Theorem 5.6] and [G, Corollary 2.15].

3. CONCLUDING REMARKS

Many generalizations of our results are possible, and involve very little new technical tools.

For example, we can introduce a notion relative to subspaces.

Definition 3.1. Let I be a set, E be a subset of $\mathcal{S}(I)$, and Y be a subspace of some topological space X .

We say that Y *satisfies the E -accumulation property in X* if and only if, for every I -indexed sequence $(y_i)_{i \in I}$ of elements of Y , there is $x \in X$ such that, in X , x is an accumulation point of $(y_i)_{i \in I}$ in the sense of E .

If we take $Y = X$, we reduce to the case dealt in the last statement of Definition 1.1.

We now show that, for “small” E , the E -accumulation property is trivially false, for T_1 spaces with more than one point.

Proposition 3.2. *Suppose that I is a set, $E \subseteq \mathcal{S}(I)$, and there is a set $J \subseteq I$ such that neither J nor $I \setminus J$ belong to E . In particular, this assumption applies when E is a filter on I which is not an ultrafilter.*

If some topological space X satisfies the E -accumulation property, then, for every pair x, y of distinct points of X , there is some point z of X such that every neighborhood U of z contains both x and y .

Proof. Suppose that $x \neq y \in X$. Consider the sequence defined by $x_i = x$, if $i \in J$, and $x_i = y$, if $i \in I \setminus J$, and apply the E -accumulation property to this sequence. \square

In particular, if X is a T_1 topological space, $|X| > 1$ and X satisfies CAP_α^* , then $\alpha = \omega^\eta$ (ordinal exponentiation), for some ordinal η . Otherwise (for example, expressing α in normal form) α is the union of two disjoint subsets each of order type $< \alpha$, and we can apply Proposition 3.2.

Thus, (full) compactness does not necessarily imply CAP_α^* , for every ordinal α , though it is well known that it implies CAP_λ , for every infinite cardinal λ .

We can also give a two-ordinals generalization of Definition 2.5.

Definition 3.3. If β, α are ordinals, X is a topological space, and $(x_\gamma)_{\gamma \in \alpha}$ is a sequence of elements of X , we say that $x \in X$ is a β - α -complete accumulation point of $(x_\gamma)_{\gamma \in \alpha}$ if and only if, for every neighborhood U of x , the set $\{\gamma \in \alpha \mid x_\gamma \in U\}$ has order type $\geq \beta$.

We say that X satisfies the β - α -complete accumulation property, if and only if every α -indexed sequence $(x_\gamma)_{\gamma \in \alpha}$ of elements of X has some β - α -complete accumulation point in X .

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